Worcester County Mathematics League

Varsity Meet 1 - October 11, 2023

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League
Varsity Meet 1 - October 11, 2023
Answer Key

Round 1 - Arithmetic

1. -30
2. $\frac{8}{3}$ or $2\frac{2}{3}$ or $2.\overline{6}$
3. 2

Round 2 - Algebra I

1. 5
2. 4
3. $(5, 1, 3)$ (exact order only)

Round 3 - Set Theory

1. 59
2. 9
3. 256

Round 4 - Measurement

1. 15
2. $(72, 3)$ (exact order only)
3. $(18, -18, 3)$ (exact order only)

Round 5 - Polynomial Equations

1. $3 + \sqrt{2}, 3 - \sqrt{2}$ (need both, either order)
2. 20
3. $u = 6+a, v = 9+3a+b$ (sum order doesn't matter)

Team Round

1. 24
2. $(51, 15)$ (exact order)
3. 90
4. $(15, -10, 2)$ (exact order)
5. -6
6. 2
7. 10
8. 2π
9. $(-5, 2, -10)$ (exact order)

Worcester County Mathematics League
Varsity Meet 1 - October 11, 2023
Round 1 - Arithmetic

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Evaluate the expression shown below for $y = 2$:

$$y - y \cdot y^{y^2}$$

2. Let $a * b = \dfrac{a + \frac{1}{b}}{b + \frac{1}{a}}$. Find $2 * (3 * 4)$.

3. Evaluate the expression shown below:

$$(-2)^4 - 3\,|5 - 11| + 2\left(\frac{2(7) - 4\,|2 - 3^0|}{\sqrt{-4^2 + 6 \cdot 8 - 7}}\right)$$

ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. ______________________

(3 pts) 3. ______________________

Westborough, Mass. Academy, Ashland

Worcester County Mathematics League

Varsity Meet 1 - October 11, 2023

Round 2 - Algebra I

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Solve for x:

$$\frac{x}{5} + \frac{x}{10} + \frac{x}{15} + \frac{x}{20} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

2. If 1 boodle is equal to 15 noodles and 1 noodle is equal to 12 poodles, how many boodles equal 720 poodles?

3. Solve the following system for r, s, and t. Express your answer as the ordered triple (r, s, t).

$$\frac{6}{t} + 3s = r$$
$$r - s = 4$$
$$\frac{t}{s} = 3$$

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

(3 pts) 3. $(r, s, t) =$ (____________)

Tahanto, St. John's, Bromfield

Worcester County Mathematics League
Varsity Meet 1 - October 11, 2023
Round 3 - Set Theory

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. In one year at Littleknown HS, 100 students played basketball and 84 played softball. In total, 125 students played at least one or both of these sports. How many students played both basketball and softball?

2. 70 people are asked whether they like chocolate, vanilla, and strawberry ice cream. 30 people like chocolate, 30 like vanilla, 25 like strawberry, 7 like chocolate and vanilla, 12 like chocolate and strawberry, 8 like none of these flavors, 5 like all three flavors. How many people like vanilla and strawberry?

3. Let $S = \{n : n \text{ is a positive multiple of 2 less than 50}\}$.
   Let $T = \{n : n \text{ is a positive multiple of 3 less than 100}\}$.
   How many subsets of S are also subsets of T?

ANSWERS

(1 pt) 1. ________________

(2 pts) 2. ________________

(3 pts) 3. ________________

West Boylston, Blackstone Valley, Algonquin

Worcester County Mathematics League

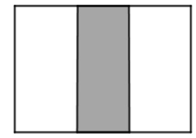
Varsity Meet 1 - October 11, 2023

Round 4 - Measurement

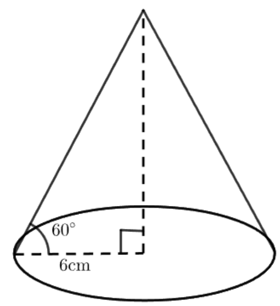
All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

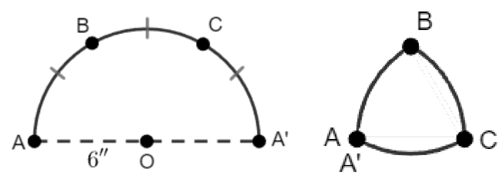
1. Two $5'X5'$ squares overlap to form a $5'X7'$ rectangle, shown at right. Find the area of the shaded region in which the two squares overlap.



2. The volume of a right circular cone with radius of 6cm can be expressed in simplest radical form as $a\sqrt{b}\,\pi$ cm^3 if the angle pictured at right has measure $60°$. Find the ordered pair (a, b).



3. A semicircle of radius 6 inches is trisected into three arcs, shown at near right as AA'. The three arcs are folded so that C coincides with O and A coincides with A', shown at far right. The area of the figure thus created (a curvilinear triangle) can be written in simplest radical form as $a\pi + b\sqrt{c}$ in^2. Find the ordered triple (a, b, c).



ANSWERS

(1 pt) 1. ______________________ ft^2

(2 pts) 2. $(a, b) =$ (______________________)

(3 pts) 3. $(a, b, c) =$ (______________________)

Burncoat, Shrewsbury, Worc. Academy

Worcester County Mathematics League

Varsity Meet 1 - October 11, 2023

Round 5 - Polynomial Equations

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Find the roots of $y^2 - 6y + 7 = 0$.

2. Find the single positive integer solution to the following equation:

$$n^2 + (n+5)^2 + (n+2)^2 = (n+7)^2 + (n+8)^2 - 4$$

3. The roots of the quadratic $x^2 + ax + b = 0$ are 3 greater than the roots of $x^2 + ux + v = 0$. Find expressions for u and v, both, in terms of a and/or b.

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

(3 pts) 3. $u =$ _______________ $v =$ _______________

Shepherd Hill, Blackstone Valley, St. John's/QSC

Worcester County Mathematics League
Varsity Meet 1 - October 11, 2023
Team Round

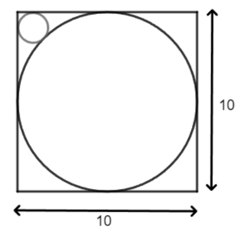
All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Operator $\odot$ is defined as $a \odot b = (a+b)^2 - (a-b)^2$. Find the value of $\sqrt{18} \odot \sqrt{2}$.

2. Let m be a two digit positive integer and n be a two digit integer equal to m with its digits reversed. If $\frac{m}{n} = \frac{17}{5}$, find the ordered pair (m, n).

3. There are 250 students at Math Rocks High School who are taking at least one of three AP courses: Calculus, Biology and English Literature. There are 160 students taking AP Calculus, 135 taking AP Biology, and 125 taking AP English Literature. There are 40 students taking all three AP courses. How many students are taking exactly two of these courses?

4. A circle is inscribed in a 10 X 10 square. Between the circle and one corner of the square, a smaller circle is inscribed so that it is tangent to the circle and two sides of the square. See the figure at right. The radius of the smaller circle is written in simplest radical form as $a + b\sqrt{c}$. Find the ordered triple (a, b, c).



5. Let $p(x) = x^5 + 2x^4 + 4x^2 + 5x + k$. If $x+2$ is a factor of $p(x)$, then find k.

6. Define $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$ (n factorial). Let $N = 15! - 10!$. The last (least significant) k digits of N are 0 (zero). Find k.

7. Let $A = \{x : 3x + 5 \leq 15\}$. Let $B = \{n : n \text{ is an integer and } |n| < 7\}$. Find $|A \cap B|$, the number of elements in $A \cap B$.

8. A belt is stretched tight about the equator of the earth. How much must the belt be lengthened, in feet, so that it can circle the earth one foot outside the equator at all points? Assume that the equator of the earth is a perfect circle.

9. Polynomial $p(x) = x^3 + ax^2 + bx + c$ with real coefficients a, b and c has roots 5 and $i\sqrt{2}$. That is, $p(5) = 0$ and $p(i\sqrt{2}) = 0$. Find the ordered triple (a, b, c).

ANSWERS

1. ________________________

2. $(m, n) = ($________________$)$

3. ________________________

4. $(a, b, c) = ($______________$)$

5. $k =$ ____________________

6. $k =$ ____________________

7. ________________________

8. ________________________ ft

9. $(a, b, c) = ($______________$)$

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2. 9
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4. $(15, -10, 2)$ (exact order)
5. -6
6. 2
7. 10
8. 2π
9. $(-5, 2, -10)$ (exact order)

Round 1 - Arithmetic

1. Evaluate the expression shown below for $y = 2$:

$$y - y \cdot y^{y^2}$$

Solution: First, substitute (four times) $y = 2$ into the expression: $2 - 2 \cdot 2^{2^2}$. Next, simplify the exponents: $2^{2^2} = 2^4 = 16$ and $2^2 = 4$, resulting in:

$$2 - 2 \cdot 16$$

Next, multiply and only then subtract the two remaining numbers: $2 - 32 = \boxed{-30}$.

Note that exponents of exponents in the absence of parentheses are evaluated "top down", or $2^{\left(2^2\right)}$ in this case. However, the order of evaluating the exponents doesn't matter for this expression because $\left(2^2\right)^2 = 4^2 = 16$.

2. Let $a * b = \dfrac{a + \frac{1}{b}}{b + \frac{1}{a}}$. Find $2 * (3 * 4)$

Solution: Before plugging in numbers, check to see whether the original expresion for $a * b$ can be simplified (it can!).

$$a * b = \frac{a + \frac{1}{b}}{b + \frac{1}{a}} = \frac{\frac{ab+1}{b}}{\frac{ab+1}{a}} = \frac{ab+1}{b} \cdot \frac{a}{ab+1} = \frac{a}{b}$$

Using this expression simplifies the calculations: $2 * (3 * 4) = 2 * \left(\dfrac{3}{4}\right) = \frac{2}{\frac{3}{4}} = 2 \cdot \frac{4}{3} = \boxed{\dfrac{8}{3}}$.

3. Evaluate the expression shown below:

$$(-2)^4 - 3\,|5 - 11| + 2\left(\frac{2(7) - 4\,|2 - 3^0|}{\sqrt{-4^2 + 6 \cdot 8 - 7}}\right)$$

Solution: Note that $(-2)^4 = 16$ (even power, therefore positive), $|5 - 11| = |-6| = 6$, $2(7) = 14$, $2 - 3^0 = 2 - 1 = 1$, $-4^2 = -16$ (apply the exponent, *then* the negative sign), and $6 \cdot 8 = 48$. Substituting these numbers into the expression leaves:

$$\begin{aligned} 16 - 3(6) + 2\left(\frac{14 - 4(1)}{\sqrt{-16 + 48 - 7}}\right) &= 16 - 18 + 2\left(\frac{10}{\sqrt{32 - 7}}\right) \\ &= -2 + \frac{20}{\sqrt{25}} = -2 + \frac{20}{5} = -2 + 4 = \boxed{2}. \end{aligned}$$

Round 2 - Algebra I

1. Solve for x:

$$\frac{x}{5}+\frac{x}{10}+\frac{x}{15}+\frac{x}{20}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$$

Solution: First, look for common factors and resist the temptation to combine fractions. On the left hand side (LHS) of the equation, x is a common factor. Looking further, $\frac{1}{5}$ is also a factor. Factor $\frac{x}{5}$ from the LHS and:

$$\frac{x}{5}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$$

Now divide both sides by $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$ *without* combining the fractions, and $\frac{x}{5}=1$, or $x=\boxed{5}$.

2. If 1 boodle is equal to 15 noodles and 1 noodle is equal to 12 poodles, how many boodles equal 720 poodles?

Solution: Start with 720 poodles, factoring poodles into equivalent noodles, then factoring noodles into equivalent boodles:

$$\begin{aligned}720 \text{ poodles} &= 60\cdot(12 \text{ poodles}) = 60\cdot(1 \text{ noodle}) = 60 \text{ noodles}\\ &= 4\cdot(15 \text{ noodles}) = 4\cdot(1 \text{ boodle}) = \boxed{4} \text{ boodles}\end{aligned}$$

3. Solve the following system for r, s, and t. Express your answer as the ordered triple (r,s,t).

$$\begin{aligned}\frac{6}{t}+3s&=r\\ r-s&=4\\ \frac{t}{s}&=3\end{aligned}$$

Solution: Proceed by the method of substitution, ending with an equation in one unknown. First, $\frac{t}{s}=3$, so $t=3s$. Substituting $t=3s$ into the first equation results in $\frac{6}{t}+t=r$. Substituting $s=\frac{t}{3}$ into the second equation results in $r-\frac{t}{3}=4$, or $r=4-\frac{t}{3}$. Now substitute $r=4-\frac{t}{3}$ into the first equation, resulting in $\frac{6}{t}+t=4+\frac{t}{3}$. Multiply both sides of the equation by $3t$: $18+3t^2=12t+t^2$. Gather terms on the right hand side: $3t^2-t^2-12t+18=2t^2-12t+18=0$. Divide by 2, resulting in a factorable quadratic: $t^2-6t+9=(t-3)^2=0$. So the quadratic has the single solution (double root) $t=3$. Then $s=\frac{t}{3}=1$ and $r-s=4$, so $r=4+s=5$. Summing up: $(r,s,t)=\boxed{(5,1,3)}$.

Round 3 - Set Theory

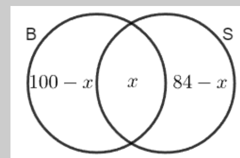
1. In one year at Littleknown High School, 100 students played basketball and 84 played softball. In total, 125 students played at least one or both of these sports. How many students played both basketball and softball?

Solution: The Principle of Inclusion-Exclusion (PIE) applied to sets A and B states:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

where $|A|$ = the number of elements in set A. It was given that $|B \cup S| = 125$, where B = students playing basketball and S = students playing softball. Also $|B| = 100$, $|S| = 84$, and $x = |B \cap S|$ (the answer to this problem). Then $125 = 100 + 84 - x$, and $x = 100 + 84 - 125 = 184 - 125 = \boxed{59}$.

This problem can also be solved using a Venn diagram depicting intersecting sets B and S, shown at right. The intersection between B and S has x elements. The regions of B and S outside the intersection have $100 - x$ and $84 - x$ elements. Then solve $(100 - x) + x + (84 - x) = 125$ for x to arrive at the same answer.



2. 70 people are asked whether they like chocolate, vanilla, and strawberry ice cream. 30 people like chocolate, 30 like vanilla, 25 like strawberry, 7 like chocolate and vanilla, 12 like chocolate and strawberry, 8 like none of these flavors, 5 like all three flavors. How many people like vanilla and strawberry?

Solution: Define the sets C = people who like chocolate, V = people who like vanilla, and S = people who like strawberry. The Principle of Inclusion-Exclusion (PIE) stated for three sets is:

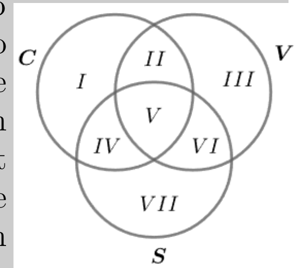
$$|C \cup V \cup S| = |C| + |V| + |S| - |C \cap V| - |C \cap S| - |V \cap S| + |C \cap V \cap S|$$

where $|A|$ is the number of elements in set A. Note that $|C \cup V \cup S| = 70 - 8 = 62$ because 8 of the 70 people like none of the three flavors. Also let $|V \cap S| = x$, the answer to the question. Plug in the given information to this equation:

$$\begin{aligned} 62 &= 30 + 30 + 25 - 7 - 12 - x + 5 \\ &= 85 - 19 - x + 5 \\ &= 66 + 5 - x \\ &= 71 - x \end{aligned}$$

and $x = 71 - 62 = \boxed{9}$.

(**alternative solution**) Instead of PIE, the problem can be solved using a Venn diagram. The seven regions of the three set Venn diagram are labeled with Roman numerals I through VII in the diagram shown at right. The first step is to write 5 in region V because that information was given. Then $|C \cap S| = 12 = |IV| + 5$, so write $12 - 5 = 7$ in region IV (here $|IV|$ stands for the number of elements/people in region IV). Likewise $|C \cap V| = 7 = |II| + 5$, so write 2 in region II. Then $|C| = 30 = |I| + 7 + 5 + 2$, so write 16 $(= 30 - 7 - 5 - 2)$ in region I. Now let $x = |S \cap V| = |V| + |VI| = 5 + |VI|$, so $|VI| = x - 5$. Then $|V| = 30$ (people who like vanilla) can be used to determine that $|III| = 28 - x$ and $|S| = 25$ can be used to determine that $|VII| = 18 - x$. Finally, all the region totals must sum to 62, so $30 + 28 - x + 18 - x + x - 5 = 62$, or $71 - x = 62$. Solve this equation to find that $x = 9$, as above.



3. Let $S = \{n : n \text{ is a positive multiple of 2 less than 50}\}$.
Let $T = \{n : n \text{ is a positive multiple of 3 less than 100}\}$.
How many subsets of S are also subsets of T?

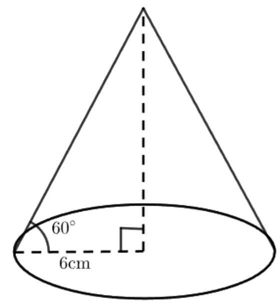
Solution: The question asks for the number of sets that are subsets of *both* S and T. Note that if R is a subset of both S and T, then every element of R must be an element of *both* S and T. Therefore, the elements of R are are elements of $S \cap T$. Thus, the question is answered by finding the number of subsets of $Q = S \cap T$. The elements of Q are positive multiples of both 2 and 3 that are less than 50 because they must satisfy the rules of both S and T. That is, they are positive multiples of $2 \cdot 3 = 6$, so $Q = \{6, 12, 18, ..., 48\}$ and Q has 8 elements. The number of subsets of a set with k elements is 2^k. (There are two choices for each element, present or not present in a given subset, and the choices are independent for each element.) In this case, $k = 8$, and $|Q| = |S \cap T| = 2^8 = \boxed{256}$.

Round 4 - Measurement

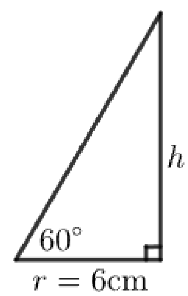
1. Two $5'X5'$ squares overlap to form a $5'X7'$ rectangle, shown at right. Find the area of the shaded region in which the two squares overlap.

Solution: The $5'X7'$ rectangle has area $5 \cdot 7 = 35\text{ft}^2$. That area is equal to the sum of the areas of the two squares ($5 \cdot 5 = 25\text{ft}^2$, each) minus the overlap area. Call the overlap area x. Then $2(25) - x = 35$, and $x = 50 - 35 = \boxed{15}\,\text{ft}^2$.

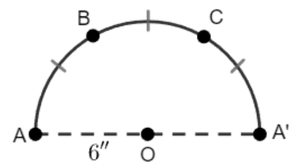
2. The volume of a right circular cone with radius of 6cm can be expressed in simplest radical form as $a\pi\sqrt{b}$ cm^3 if the angle pictured at right has measure $60°$. Find the ordered pair (a, b).

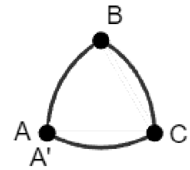


Solution: The volume of a right circular cone is given by $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height of the cone. Note the triangle with the two dashed lines in the first figure. The two sides of the triangle are the radius r and height h. This triangle, redrawn and labeled at right, is a $30° - 60° - 90°$ triangle. Recall that the sides of such a triangle are in proportion $1 : \sqrt{3} : 2$. Therefore the ratio $(r : h) = (1 : \sqrt{3})$. Write this proportion as the equality of fractions and substitutue $r = 6$: $\dfrac{6}{h} = \dfrac{1}{\sqrt{3}}$. Solving, $h = 6\sqrt{3}$. Finally, plug $r = 6$ and $h = 6\sqrt{3}$ into the formula and $V = \frac{1}{3}\pi(6^2)6\sqrt{3} = \frac{6}{3}\pi 6 \cdot 6\sqrt{3} = 2 \cdot 36\sqrt{3}\,\pi = 72\sqrt{3}\,\pi$ cm^3. Then $(a, b) = \boxed{(72, 3)}$.

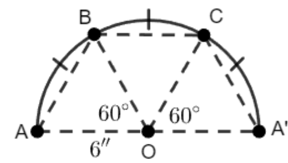


3. A semicircle of radius 6 inches is trisected into three arcs, shown at near right as AA'. The three arcs are folded so that C coincides with O and A' coincides with A, shown at far right. The area of the figure thus created (a curvilinear triangle) can be written in simplest radical form as $a\pi + b\sqrt{c}$ in^2. Find the ordered triple (a, b, c).

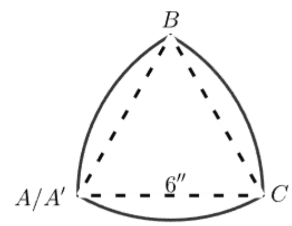




Solution: First note that the three arcs each have measure 60° because the measure of a semicircle is 180° and $180 \div 3 = 60$. Draw the radii $\overline{OB}$ and $\overline{OC}$ and the three chords $\overline{AB}$, $\overline{BC}$, and $\overline{CA'}$, shown as dotted lines in the figure at right. The central angles for the three arcs are 60°, and the three triangles formed by two radii and a chord are equilateral triangle with side length 6".



Solution: (continued) Note that the 6" chords are preserved when the arcs are folded, so that ΔABC, inside the curvilinear triangle, is an equilateral triangle with 6" sides, as shown at right. The curvilinear triangle can be formed by overlaying the three 60° sectors with the centers of the sectors located at A, B, and C. The area of the curvilinear triangle is therefore equal to the sum of the areas of the three sectors minus twice the area of overlap, which is ΔABC. The area of the semicircle is equal to $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi 6^2 = 18\pi$ in^2, and is equal to the sum of the areas of the three sectors. Thus, if A is the area of the curvilinear triangle, then:



$$A = 18\pi - 2(\text{Area of equilateral triangle with } 6'' \text{ sides})$$
$$= 18\pi - 2\left(\frac{\sqrt{3}}{4}6^2\right) = 18\pi - 2\sqrt{3}\cdot\frac{6}{2}\cdot\frac{6}{2} = 18\pi - 2\cdot 9\sqrt{3} = 18\pi - 18\sqrt{3}$$

after applying the formula for the area of an equilateral triangle of side length s ($\frac{\sqrt{3}}{4}s^2$). Thus, $(a, b, c) = \boxed{(18, -18, 3)}$.

Round 5 - Polynomial Equations

1. Find the roots of $y^2 + 6y + 7 = 0$.

Solution: A quick check verifies that this polynomial doesn't factor over the integers. Therefore recall the quadratic formula. For a quadratic equation in standard form, $ax^2 + bx + c = 0$, the roots are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In this case, $a = 1$, $b = 6$, and $c = 7$. Plug these numbers into the formula and simplify:

$$\begin{aligned} x &= \frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 7}}{2 \cdot 1} \\ &= \frac{-6 \pm \sqrt{36 - 28}}{2} = -3 \pm \frac{\sqrt{8}}{2} \\ &= -3 \pm \frac{\sqrt{4}\sqrt{2}}{2} = -3 \pm \frac{2\sqrt{2}}{2} = -3 \pm \sqrt{2} \end{aligned}$$

And the roots are $\boxed{-3 + \sqrt{2}, 3 - \sqrt{2}}$.

Other methods to solve quadratics include completing the square.

2. Find the single positive integer solution to the following equation:

$$n^2 + (n+5)^2 + (n+2)^2 = (n+7)^2 + (n+8)^2 - 4$$

Solution: Expand each of the squares using the identity $(a+b)^2 = a^2 + 2ab + b^2$, then add like terms for both the left hand side (LHS) and the right hand side (RHS) of the equation:

$$\begin{array}{ccccccccccc} n^2 & & & & & = & n^2 & + & 14n & +49 \\ n^2 & + & 10n & + & 25 & = & n^2 & + & 16n & +64 \\ n^2 & + & 4n & + & 4 & = & & & & -4 \\ \hline 3n^2 & + & 14n & + & 29 & = & 2n^2 & + & 30n & +109 \end{array}$$

Next, subtract the RHS from the LHS to put the quadratic equation in standard form:

$$\begin{array}{ccccc} 3n^2 & + & 14n & + & 29 \\ -(2n^2 & + & 30n & + & 109) \\ \hline n^2 & - & 16n & - & 80 \end{array}$$

Note that $-80 = -20 \cdot 4$ and $-20 + 4 = -16$, so that this quadratics factors into $(n - 20)(n + 4)$. The solutions are 20 and -4. The only positive solution is $\boxed{20}$.

3. The roots of the quadratic $x^2 + ax + b = 0$ are 3 greater than the roots of $x^2 + ux + v = 0$. Find expressions for u and v, both, in terms of a and/or b.

Solution: Substitute $r + 3$ into the first equation, noting that if $r + 3$ is a root of the first equation, then r will be a root of $x^2 + ux + v = 0$. Expand the polynomial in r and write it in standard form:

$$(r+3)^2 + a(r+3) + b = 0$$
$$r^2 + ry + 9 + ay + 3a + b = 0$$
$$r^2 + (6+a)r + 3a + b + 9 = 0$$

Now, if $r + 3$ is a root of the first equation, then r will be a root of this equation when $6 + a = u$ and $3a + b + 9 = v$. Therefore, $\boxed{u = 6 + a}$ and $\boxed{v = 9 + 3a + b}$.

Viete's Theorem can be also be used to arrive at this result. The roots of the first equation sum to $-a$. The roots of the second equation sum to $-u$. Since there are two roots to each equation, the second sum must be 6 less than the first sum. Therefore, $-u = -a - 6$, and $u = 6 + a$ (check).

The product of the roots of the first equation, $r_1 \cdot r_2$, is b, or $r_1 \cdot r_2 = b$. The product of the roots of the second equation $(r_1 - 3)(r_2 - 3)$ is v, or $r_1 \cdot r_2 - 3r_1 - 3r_2 + 9 = v$. Rewrite this equation, substituting $r_1 \cdot r_2 = b$: $b - 3(r_1 + r_2) + 9 = v$. Now substitute $r_1 + r_2 = -a$ and: $b + 3a + 9 = v$ (check).

Team Round

1. Operator $\odot$ is defined as $a \odot b = (a+b)^2 - (a-b)^2$. Find the value of $\sqrt{18} \odot \sqrt{2}$.

Solution: This problem is most quickly solved by first simplifying the expression for $a \odot b$:

$$\begin{aligned} a \odot b = (a+b)^2 - (a-b)^2 &= a^2 + 2ab + b^2 - \left(a^2 - 2ab + b^2\right) \\ &= a^2 - a^2 + 2ab + 2ab + b^2 - b^2 \\ &= 4ab \end{aligned}$$

Now plug is $a = \sqrt{18}$ and $b = \sqrt{2}$: $\sqrt{18} \odot \sqrt{2} = 4 \cdot \sqrt{18} \cdot \sqrt{2} = 4 \cdot \sqrt{18 \cdot 2} = 4 \cdot \sqrt{36} = 4 \cdot 6 = \boxed{24}$.

2. Let m be a two digit positive integer and n be a two digit integer equal to m with its digits reversed. If $\frac{m}{n} = \frac{17}{5}$, find the ordered pair (m, n).

Solution: Because m and n are positive integers, then there is some integer k such that $\frac{m}{n} = \frac{17k}{5k}$. Note that $k \geq 2$ for $5k$ to be a two digit number ($5k \geq 10$). Also, note that $k < 6$ because $17k < 100 < 102 = 17 \cdot 6$. The answer can be found by trial and error since there k must be one of four numbers $(2, 3, 4, 5)$. When $k = 3$, $m = 3 \cdot 17 = 51$ and $n = 3 \cdot 5 = 15$, and $(m, n) = \boxed{(51, 15)}$.

3. There are 250 students at Math Rocks High School who are taking at least one of three AP courses: Calculus, Biology and English Literature. There are 160 students taking AP Calculus, 135 taking AP Biology, and 125 taking AP English Literature. There are 40 students taking all three AP courses. How many students are taking exactly two of these courses?

Solution: The 250 students can be divided into three exclusive sets:

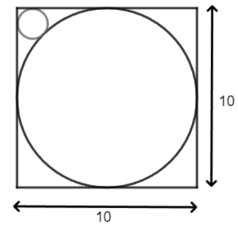
$$\begin{aligned} Q_1 &= \text{the number of students taking exactly one course} \\ Q_2 &= \text{the number of students taking exactly two courses} \\ Q_3 &= \text{the number of students taking all three courses (Calculus, Biology, English)} \end{aligned}$$

Then $Q_1 + Q_2 + Q_3 = 250$. Adding the given numbers of students taking Calculus, Biology, and English overcounts the total number of students. Specifically, each student taking exactly two courses is counted twice and each student taking all three courses is counted three times, or $Q_1 + 2Q_2 + 3Q_3 = 160 + 135 + 125 = 420$. Subtract the first equation from the second equation:

$$\begin{array}{ccccccc} Q_1 & + & 2Q_2 & + & 3Q_3 & = & 420 \\ -(Q_1 & + & Q_2 & + & Q_3 & = & 250) \\ \hline & & Q_2 & + & 2Q_3 & = & 170 \end{array}$$

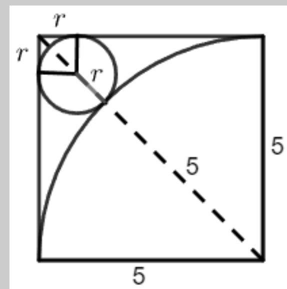
Note that $Q_3 = 40$ is given, and that the problem asks for Q_2. Therefore $Q_2 = 170 - 2(40) = 170 - 80 = \boxed{90}$.

4. A circle is inscribed in a 10 X 10 square. Between the circle and one corner of the square, a smaller circle is inscribed so that it is tangent to the circle and two sides of the square. See the figure at right. The radius of the smaller circle is written in simplest radical form as $a + b\sqrt{c}$. Find the ordered triple (a, b, c).



Solution: The larger circle is tangent to the square at the midpoints of the four sides of the square. The line segments that join opposite midpoints are diameters of the circle and have length 10. Therefore the radius of the larger circle is 5. The line sements divide the the square into quarters, each a $5X5$ square. Draw the quarter square with the smaller inscribed circle.

Next, draw the diagonal of the square that passes through the center of the smaller inscribed circle, shown at right. Call the radius of that circle r. Draw the two radii of this circle to the points of tangency with the square, as shown, forming a smaller square of side length r.



Now, consider the diagonal of the larger square. Its length is $5\sqrt{2}$. The diagonal of the smaller square is $r\sqrt{2}$, and it lies on the longer diagonal. In fact, the longer diagonal is divided into three segments by the point of tangency of the two circles and the center of the smaller circle. The sum of these three lengths, $r\sqrt{2}$, r, and $5\sqrt{2}$ (listed from top left to bottom right) is equal to the length of the larger square diagonal $(5\sqrt{2})$:

$$r\sqrt{2} + r + 5 = 5\sqrt{2}$$

Subtract 5 from both sides and factor:

$$r(1 + \sqrt{2}) = 5(\sqrt{2} - 1)$$

Now $r = 5\dfrac{\sqrt{2}-1}{\sqrt{2}+1} = 5\dfrac{(\sqrt{2}-1)^2}{(\sqrt{2}+1)(\sqrt{2}-1)} = 5\dfrac{2 - 2\sqrt{2} + 1}{(\sqrt{2})^2 - 1} = 5\dfrac{3 - \sqrt{2}}{1} = 15 - 10\sqrt{2}$.

Therefore $(a, b, c) = \boxed{(15, -10, 2)}$.

5. Let $p(x) = x^5 + 2x^4 + 4x^2 + 5x + k$. If $x + 2$ is a factor of $p(x)$, then find k.

Solution: By the Factor Theorem, if $x + 2$ is a factor, then $p(-2) = 0$. Then k can be found by substituting $x = -2$ in $p(x) = 0$, and solving for k. The method of synthetic substitution is usually quicker and less error prone, and we will use that method here:

$$\begin{array}{r|rrrrrr} -2 & 1 & 2 & 0 & 4 & 5 & k \\ & & -2 & 0 & 0 & -8 & 6 \\ \hline & 1 & 0 & 0 & 4 & -3 & | \; k+6 \end{array}$$

So $p(-2) = k + 6 = 0$ and $k = \boxed{-6}$, ensuring that $x + 2$ is a factor of $p(x)$.

6. Define $n! = n \cdot (n-1) \cdot (n-2) \cdot ... \cdot 2 \cdot 1$ (n factorial). Let $N = 15! - 10!$. The last (least significant) k digits of N are 0 (zero). Find k.

Solution: If 10^k divides N, then the last digits of N will be 0 (zero). For 10^k to divide N, 2^k and 5^k must each divide N. The method therefore is to count the factors of 2 and 5, and k will be the smaller of those two numbers.

To start, factor N, turning it from a sum expression into a product expression:

$$\begin{aligned} N = 15! - 10! &= 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot (10!) - 10! \\ &= (15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 - 1)(10!) \end{aligned}$$

Now 10! has two factors of 5 because 10 and 5 are factors of 10!. It has more factors of 2 (eight, in total). Therefore 10^2 is a factor of 10!. Next, note that 5 is a factor of $15 \cdot 14 \cdot 13 \cdot 12 \cdot 11$. Therefore, 5 cannot be a factor of $15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 - 1$, so that N has only two factors of 5. Thus, N has exactly 2 factors of 10, so the last $\boxed{2}$ digits of N are zeroes.

7. Let $A = \{x : 3x + 5 \leq 15\}$. Let $B = \{n : n \text{ is an integer and } |n| < 7\}$. Find $|A \cap B|$, the number of elements in $A \cap B$.

Solution: Starting with A, $3x + 5 \leq 15$, so $3x \leq 10$, and $x \leq \frac{10}{3}$. For x to be an element of $A \cap B$, it must be an element of *both* A and B. So, x must be an integer, x must satisfy $-7 < x < 7$, and x must satisfy $x \leq \frac{10}{3}$. Then:

$$A \cap B = \{x : x \text{ is an integer and } -7 < x \leq \frac{10}{3}\}$$

Specifying by list:

$$A \cap B = \{-6, -5, -4, ..., 1, 2, 3\}$$

and $|A \cap B| = \boxed{10}$.

8. A belt is stretched tight about the equator of the earth. How much must the belt be lengthened, in feet, so that it can circle the earth one foot outside the equator at all points? Assume that the equator of the earth is a perfect circle.

Solution: Surprisingly, the answer does *not* depend on the circumference of the earth. Let $R =$ the radius of the earth. Then the length of the belt is the circumference of the earth, or $2\pi R$. If the belt is one foot outside of the equator, then it forms a circle of radius $R + 1$ feet. The new length of the belt is $2\pi(R + 1)$ feet, the circumference of this (slightly) larger circle. The increase in length is:

$$2\pi(R + 1) - 2\pi R == 2\pi R + 2\pi - 2\pi R = \boxed{2\pi} \text{ feet}$$

9. Polynomial $p(x) = x^3 + ax^2 + bx + c$ with real coefficients a, b and c has roots 5 and $i\sqrt{2}$. That is, $p(5) = 0$ and $p(i\sqrt{2}) = 0$. Find the ordered triple (a, b, c).

Solution: By the complex conjugate root theorem, if $i\sqrt{2}$ is a root of $p(x)$, then $-i\sqrt{2}$ is also a root. Therefore $p(x)$ has factors $(x - i\sqrt{2})$ and $(x + i\sqrt{2})$. Note that $(x - i\sqrt{2})(x + i\sqrt{2}) = x^2 - (i\sqrt{2})^2 = x^2 - (-2) = x^2 + 2$ is a factor of $p(x)$. Therefore $p(x) = (x^2 + 2)(x - 5) = x^3 - 5x^2 + 2x - 10$ and $(a, b, c) = \boxed{(-5, 2, -10)}$